

Here is another proof that $M_{f,C}$ is simple when f is irreducible.

With $wI_C := \text{span}\{pq^* \in L(\Gamma) \mid p, q \in P(\Gamma), sp = w, \exists N \in \mathbb{N} \text{ s.t. } (C^*)^N p = 0\}$

Consider the diagram:

$$\begin{array}{ccccccc}
& & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \\
0 & \longrightarrow & wI_C & \xrightarrow{f(C^*)_-} & wI_C & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \\
0 & \longrightarrow & wL(\Gamma) & \xrightarrow{f(C^*)_-} & wL(\Gamma) & \xrightarrow{(1 \otimes w)_-} & M_{f,C} \longrightarrow 0 \\
& & \downarrow \text{ev} & & \downarrow \text{ev} & & \downarrow \text{id}_M \\
0 & \longrightarrow & M_C & \xrightarrow{f(x) \otimes \text{id}} & M_C & \xrightarrow{q} & M_{f,C} \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
& & 0 & & 0 & & 0
\end{array}$$

where q is the map that takes the quotient of M_C by the ideal $(f(x) \otimes w)$ and $\text{ev} : \alpha \mapsto (1 \otimes w)\alpha$, where the action is defined in the same way that $wL(\Gamma)$ acts on M_C .

We wish to show that the middle sequence is exact, thus giving us another description of $M_{C,f}$.

Firstly, we will show that top horizontal sequence is exact.

The map that is multiplication on the left by $\frac{1}{f(C^*)_-}$ is the inverse of $f(C^*)_-$. $\frac{1}{f(C^*)_-}$ is a power series inverse of $f(x)$ with the variable x replaced formally by C^* . Thus, for each path p in wI_C there exists a finite power N_p beyond which $(C^*)^{N_p}$ is the zero map by the definition of wI_C . Thus the top row of our diagram has the same object (wI_C) with an isomorphism bewtween, so it is exact.

The bottom row is exact the definition of $M_{f,C}$, as the image of $f(x) \otimes \text{id}_-$ is $(f(x) \otimes w)$.

The left and center column have $wI_C \subseteq \ker(\text{ev})$. For all α in $wL(\Gamma)$, we have $(1 \otimes w)\alpha = (1 \otimes w)C^N(C^*)^N\alpha$ for all N . As mutiplication is associative and $(1 \otimes w)C^N(C^*)^N = (1 \otimes w)$, if $\alpha \in \ker(\text{ev})$, then there exists an N where $(C^*)^N\alpha = 0$. This is by definition wI_C .

The left and center column have $wI_C \subseteq \ker(\text{ev})$. Consider

$$\mathcal{S} = \text{span}(\{C^k q^* \mid q \in P_C, k \in \mathbb{Z}^+\} \sqcup \{q^* \mid q \in P_C\} \sqcup \{(C^*)^k q^* \mid q \in P_C, k \in \mathbb{Z}^+\}).$$

We wish to show that $wI_C \oplus \mathcal{S} = wL(\Gamma)$. Clearly $wI_C \cap \mathcal{S} = 0$. For any path pq^* not in wI_C , $(C^*)^N p \neq 0$ for all $N \in \mathbb{N}$. This means that $p = C^k$ or $p = C^k c_0 c_1 \cdots c_i$ where $k \in \mathbb{N}$, $i \in \{1, 2, \dots, n-1\}$. When $p = C^k c_0 c_1 \cdots c_i$ where $k \in \mathbb{N}$, $i \in \{1, 2, \dots, n-1\}$, by CK2 we get $pq^* = p(\sum_{e \in s c_{i+1}} ee^*)q^* = C^k c_0 c_1 \cdots c_{i+1} c_{i+1}^* q^* + x$ where $x \in wI_C$. One continues in this manner and gets $pq^* = C^k c_{i+1}^* \cdots c_n^* q^* + x'$, where $x' \in wI_C$. Now we only need consider paths $C^k q^*$ where $k \in \mathbb{N}$. Notice by repeated application of CK2,

$$\begin{aligned} CC^* &= c_0 c_1 \cdots c_n c_n^* c_{n-1}^* \cdots c_0^* = c_0 c_1 \cdots c_{n-1} (s c_n - \sum_{\substack{s e = s c_n, \\ e \neq c_n}} ee^*) c_{n-1}^* \cdots c_1^* c_0^* \\ &= c_0 c_1 \cdots c_{n-2} c_{n-1} c_{n-1}^* c_{n-2}^* \cdots c_0^* - \sum_{\substack{s e = s c_n, \\ e \neq c_n}} c_0 c_1 \cdots c_n e e^* c_n^* c_{n-1}^* \cdots c_0^* \\ &= c_0 c_1 \cdots c_{n-3} c_{n-2} c_{n-2}^* c_{n-3}^* \cdots c_0^* - \sum_{\substack{s e = s c_{n-1}, \\ e \neq c_{n-1}}} c_0 c_1 \cdots c_{n-3} c_{n-2} e e^* c_{n-2}^* c_{n-3}^* \cdots c_0^* \\ &\quad - \sum_{\substack{s e = s c_n, \\ e \neq c_n}} c_0 c_1 \cdots c_n e e^* c_n^* c_{n-1}^* \cdots c_0^* \\ &\quad \vdots \\ &= w + \sum_{m=1}^N p_m q_m^* \end{aligned}$$

where $p_m q_m^*$ is a monomial in wI_C . Also wI_C is closed under multiplication by C^k on the left and multiplication by $e^* \in E^*$ on the right. Thus

$$\begin{aligned} C^j (C^*)^j &= C^{j-1} (w + \sum_{m=1}^N p_m q_m^*) (C^*)^{j-1} \\ &= C^{j-2} (w + \sum_{m=1}^N p_m q_m^* + \sum_{m=1}^N C p_m q_m^* C^*) (C^*)^{j-2} \\ &\quad \vdots \\ &= w + \sum_{m=1}^{N'} p'_m q'_m{}^* \end{aligned}$$

where again, $p'_m q'_m{}^*$ is a monomial in wI_C . So if $C^k (C^*)^j q^*$ where $k \in \mathbb{N}$ is st $k \neq 0$ and $q \in P_C$, then

$$C^k (C^*)^j q^* = \begin{cases} C^{k-j} q^* + \sum_{m=1}^{N'} C^{k-j} p_m q_m^* q^* & p_m q_m^* \in wI_C \ k \geq j \\ (C^*)^{j-k} q^* + \sum_{m=1}^{N'} p_m q_m^* (C^*)^{j-k} q^* & p_m q_m^* \in wI_C \ k < j \end{cases}$$

Therefore, $wL(\Gamma) = wI_C \oplus \mathcal{S}$. As $ev(\mathcal{S}) = \{x^k \otimes q \mid k \in \mathbb{Z}, q \in P_C\}$, a linearly independent set in M . So $wI_C = \ker(ev)$.

The last column is trivially exact.

We will now check that the diagram is commutative.

For the top left square: $p \mapsto f(C^*)p \mapsto f(C^*)p$, and $p \mapsto p \mapsto f(C^*)p$.

For the top right square: $p \mapsto 0 \mapsto 0$, and $p \mapsto p \mapsto 0$ (as $p \in wI_C$).

For the bottom left square: $p \mapsto (1 \otimes w)p \mapsto (f(x) \otimes w)p$, and $p \mapsto f(C^*)p \mapsto (1 \otimes w)f(C^*)p = (f(x) \otimes w)p$.

For the bottom right square: $p \mapsto (1 \otimes w)p \mapsto (1 \otimes w)p$, and $p \mapsto (1 \otimes w)p \mapsto (1 \otimes w)p$.